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Monic Polynomials with Minimal Norm

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I. INTRODUCTION

Let Π_n denote the family of monic polynomials of degree n,

$$x^n + a_1 x^{n-1} + \cdots + a_n.$$

For any real $p \geqslant 1$ we seek $f_n(x)$, the member of Π_n with least L_p norm over $(-1, 1)$, i.e., such that

$$\left\{\int_{-1}^{1} |f_n(x)|^p \, dx\right\}^{1/p} \leqslant \left\{\int_{-1}^{1} |\phi(x)|^p \, dx\right\}^{1/p}$$

for every member $\phi(x)$ of Π_n.

The solution of this problem for $p = 1, 2, \infty$ is well known. Suitably normalized, these are

p	$f_n(x)$
1	$U_n(x) =$ Tchebycheff polynomial of the second kind
2	$P_n(x) =$ Legendre polynomial
∞	$T_n(x) =$ Tchebycheff polynomial of the first kind

All three are, in fact, ultraspherical polynomials, i.e., sets of polynomials which are mutually orthogonal on $(-1, 1)$ with a weight function of the form $(1 - x^2)^{\alpha}$. Moreover, the values of α corresponding to the three cases are $1/2$, 0, $-1/2$, respectively, i.e., $\alpha = 1/p - 1/2$. It was tempting to conjecture that the same might hold for all $p \geqslant 1$. However, examination of some special cases quickly refuted this conjecture. This raised the question of whether the extremal polynomials, for some or all values of p, other than

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1, 2, ∞ might be orthogonal with some other weight functions. We still have no final answer to this question, but shall try to show in this paper how we convinced ourselves that the answer, when given, will be negative, i.e., that no such orthogonality holds for any other values of p. This conclusion is based largely on numerical computation, described in Sections 4, 5 below. Indeed, one of the purposes of our paper is to give an example of how a computer may be used to provide just such conviction.

II. Properties of the Extremal Polynomials

(1) For any given $p \geqslant 1$, and all n (0, 1, 2,...) there exists an extremal polynomial $f_n^{(p)}(x) = x^n + a_1^{(p)} x^{n-1} + \cdots + a_n^{(p)}$ such that

$$\left\{ \int_{-1}^{1} |f_n^{(p)}(x)|^p \, dx \right\}^{1/p} \leqslant \left\{ \int_{-1}^{1} |\phi(x)|^p \, dx \right\}^{1/p} \tag{2.1}$$

for all $\phi(x)$ in Π_n.

(2) The extremal polynomial is unique, i.e., strict inequality holds in (2.1) unless $\phi(x) \equiv f_n^{(p)}(x)$.

(3) For each n, all the zeros of $f_n^{(p)}(x)$ are real and lie in the interval $-1 \leqslant x \leqslant 1$.

(4) The polynomials $f_n^{(p)}(x)$ are odd or even according to the parity of n.

Proof. (1), (2) Existence and uniqueness both follow immediately from classical considerations (cf [1, Chap. 1, Section 6, Chap. 6, Section 6]).

(3) This is a special case of a classical result established by Fejér [4] for a wide class of extremal problems, of which ours is a special case. (In fact, it can also be shown that the zeros of each $f_n^{(p)}(x)$ are distinct.)

(4) If $f_n(x)$ is any monic polynomial, then so also is $(-1)^n f_n(-x)$ and it has the same L_p norm over $(-1, -1)$. It follows by (2), that $(-1)^n f_n^{(p)}(-x) \equiv f_n^{(p)}(x)$.

III. Orthogonality

If the family of polynomials $f_n^{(p)}(x)$ $\{n = 0, 1, 2,...\}$ were orthogonal with any non-negative weight function, then they would have to satisfy a relation of the type

$$f_{n+1}^{(p)}(x) = [A_n^{(p)} x + B_n^{(p)}] f_n^{(p)}(x) - C_n^{(p)} f_{n-1}^{(p)}(x) \tag{3.1}$$

with constant $A_n^{(p)}$, $B_n^{(p)}$, $C_n^{(p)}$ ([3, p. 158]). Moreover it follows from II(4) and from the monocity of the $f_n^{(p)}(x)$ that $A_n^{(p)}=1$, $B_n^{(p)}=0$, i.e., $f_{n+1}^{(p)}(x)-xf_n^{(p)}(x)$ would have to be proportional to $f_{n-1}^{(p)}(x)$, as is the case for $p=1, 2, \infty$. We have not proved that it does not hold for any other p, but shall describe here how we convinced ourselves that this is indeed the case.

For any given $p \geqslant 1$, suppose that

$$f_2^{(p)}(x)=x^2-a,$$
$$f_3^{(p)}(x)=x^3-bx,$$
$$f_4^{(p)}=x^4-cx^2+d.$$

The recurrence relation (3.1) would require that

$$f_4^{(p)}(x)-xf_3^{(p)}(x)=kf_2^{(p)}(x), \tag{3.2}$$

where k is a suitable constant. But the left-hand side of (3.2) is $(b-c)\,x^2+d$ and so $k=b-c$. We thus get

$$(b-c)\,x^2+d\equiv(b-c)(x^2-a)$$

and hence

$$a+\frac{d}{b-c}=0. \tag{3.3}$$

Our first step was to determine a, b, c, d numerically for values of p and for every such value, to calculate

$$R_p=a+\frac{d}{b-c}. \tag{3.4}$$

Extending this idea we also evaluated numerically e, f, g, h, j, where

$$f_5^{(p)}(x)=x^5-ex^3+fx,$$
$$f_6^{(p)}(x)=x^6-gx^4+hx^2-j.$$

For orthogonality $f_5^{(p)}(x)-xf_4^{(p)}(x)$ would have to be proportional to $f_3^{(p)}(x)$, and $f_6^{(p)}(x)-xf_5^{(p)}(x)$ to $f_4^{(p)}(x)$. This would lead to the relations

$$\begin{aligned} b-\frac{f-d}{e-c}&=0,\\ c-\frac{h-f}{g-e}&=0,\\ d-\frac{j}{g-e}&=0. \end{aligned} \tag{3.5}$$

We therefore evaluated also, for a range of values of p,

$$\begin{aligned} S_p &= b - \frac{f-d}{e-c}, \\ T_p &= c - \frac{h-f}{g-e}, \\ U_p &= d - \frac{j}{g-e}. \end{aligned} \tag{3.6}$$

Values of R_p, S_p, T_p, U_p as functions of p are given in Table I and as graphs in Fig. 1. They show quite clearly how all four functions vanish at $p = 1, 2$, and ∞, but nowhere else. We note, incidentally, that all four of these functions have their respective maximum and minimum at closely neighbouring values of p. In fact, the computation of the coefficients of $f_n^{(p)}(x)$ and of the resulting "discrepancy functions of p" was continued up to $n = 7$, for $1 \leqslant p \leqslant 50$. These show exactly the same behaviour as R_p, S_p, T_p, U_p but we have not included them either in Table I or in Fig. 1.

TABLE I

p	$R_p \times 10^3$	$S_p \times 10^3$	$T_p \times 10^4$	$U_p \times 10^4$
1	0	0	0	0
1.2	016	0.12	0.15	0.84
1.4	0.19	0.14	0.18	0.93
1.5	0.18	0.13	0.17	0.84
1.6	0.16	0.11	0.14	0.71
1.8	0.086	0.056	0.076	0.36
2.0	0	0	0	0
2.5	−0.23	−0.14	−0.19	−0.84
3.0	−0.43	−0.24	−0.34	−1.49
5.0	−0.91	−0.47	−0.67	−2.73
7.5	−1.12	−0.54	−0.78	−3.08
10.0	−1.17	0.55	−0.79	−3.07
12.5	−1.17	−0.54	−0.77	−2.96
15.0	−1.14	−0.52	−0.74	−2.87
20.0	−1.07	−0.47	−0.68	−2.57
25.0	−1.00	−0.43	−0.62	−2.34
30.0	−0.93	−0.40	−0.57	−2.15
40.0	−0.82	−0.35	−0.49	−1.86
50.0	−0.73	−0.31	−0.42	−1.60
60.0	−0.67			
70.0	−0.61			
80.0	−0.57			

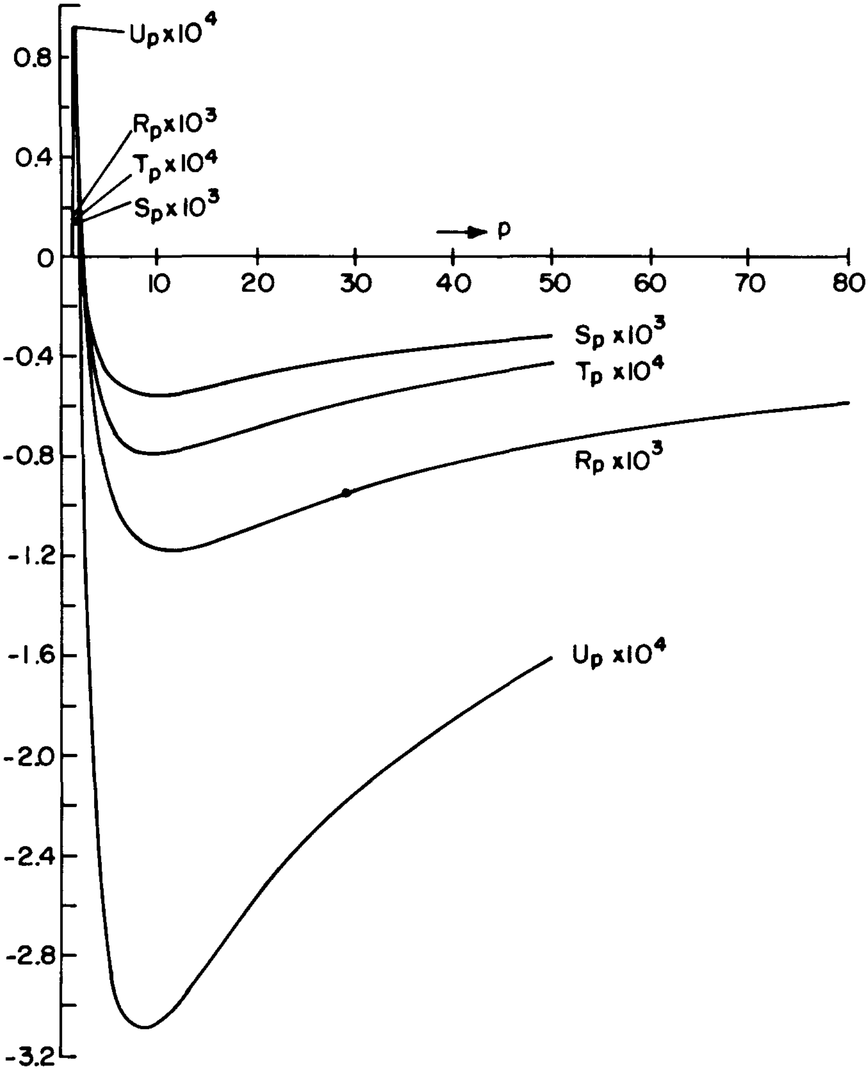

FIG. 1. Values of R_p, S_p, T_p, and U_p as functions of p.

IV. Asymptotic Estimates for Large p

The computation described in Section III and the resulting graphs in Fig. 1 do not, of course, prove that there are no other values p for which the $f_n^{(p)}(x)$ are orthogonal. Indeed we have not proved anything which could rule out the possibility of further values of p where all functions of (3.4) and (3.6) vanish simultaneously. All that we assert here is our own belief that this is not the case.

It is certainly true that $R_p < 0$ for all sufficiently large finite p. Indeed we show below that, for such p,

$$R_p \sim -\frac{1}{48p}\{\log p + \log \pi + 33 \log 2 - 24 \log 3\}$$

$$\sim -\frac{1}{48p}\{\log p - 2.3481\}. \tag{4.1}$$

We can see from Table II how this approximates the actual value of R_p as determined by numerical methods. To derive (4.1) we had first to estimate a, b, c, d for large p. Defining

$$F_p(a) = \int_0^{\sqrt{a}} (a - x^2)^p \, dx + \int_{\sqrt{a}}^1 (x^2 - a)^p dx, \tag{4.2}$$

$$G_p(b) = \int_0^{\sqrt{b}} x^p (b - x^2)^p \, dx + \int_{\sqrt{b}}^1 x^p (x^2 - b)^p dx, \tag{4.3}$$

$$H_p(\alpha, \beta) = \int_0^{\alpha} (\alpha^2 - x^2)^p (\beta^2 - x^2)^p \, dx$$

$$+ \int_{\alpha}^{\beta} (x^2 - \alpha^2)^p (\beta^2 - x^2)^p \, dx + \int_{\beta}^{1} (x^2 - \alpha^2)^p (x^2 - \beta^2)^p \, dx, \tag{4.4}$$

we estimated these functions for large p by the Laplace method ([2, Chap. 5]). This led to

$$F_p(a) \sim \frac{1}{2} a^{p+1} \sqrt{\frac{\pi}{p}} + \frac{(1-a)^{p+1}}{2p}, \tag{4.5}$$

$$G_p(b) \sim \frac{2^{p+1/2} b^{(3p+1)/2}}{3^{3p/2+1}} \sqrt{\frac{\pi}{p}} + \frac{(1-b)^{p+1}}{p(3-b)}, \tag{4.6}$$

TABLE II

p	Actual value	Approximation (4.1)	Error (%)
30	-0.928×10^{-3}	-0.731×10^{-3}	21
40	-0.818×10^{-3}	-0.698×10^{-3}	15
50	-0.733×10^{-3}	-0.651×10^{-3}	11
60	-0.665×10^{-3}	-0.606×10^{-3}	9
70	-0.611×10^{-3}	-0.566×10^{-3}	7
80	-0.566×10^{-3}	-0.530×10^{-3}	6

$$H_p \sim \frac{1}{2} c^{-1/2} d^{p+1/2} \sqrt{\frac{\pi}{p}} + \frac{(c^2-4d)^{p+1/2} c^{-1/2}}{2^{2p+3/2}} \sqrt{\frac{\pi}{p}} + \frac{(1-c+d)^{p+1}}{2p(2-c)}, \tag{4.7}$$

where $c = \alpha^2 + \beta^2$, $d = \alpha^2 \beta^2$.

The next step was to determine a, b, c, d so as to minimize F_p, G_p, and H_p, i.e., to solve the equations $dF_p/da = dG_p/db = \partial H_p/\partial c = \partial H_p/\partial d = 0$. The analysis was rather complicated but led, in the end, to

$$\begin{aligned} a &\sim 0.5 - \frac{1}{8p}\log(2\pi p), \\ b &\sim 0.75 - \frac{1}{12p}\log\left(\frac{27\pi p}{8}\right), \\ c &\sim 1 - \frac{1}{16p}\log(8\pi p), \\ d &\sim 0.125 - \frac{1}{64p}\log(32\pi p). \end{aligned} \tag{4.8}$$

Substituting these in (3.4) we get (4.1).

Unfortunately, the asymptotic estimation of $e, \ldots, j$ proved prohibitively complicated and we have contented ourselves, for the time being, with the evidence of R_p.

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